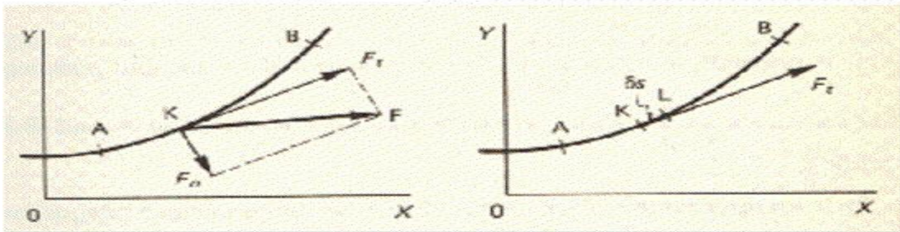
LECTURE NO 8 & 9

Line Integrals



The work done in moving the particle through a small distance s from K to L along the curve is then approximately F_1 s. So the total work done in moving a particle along the curve from A to B is given by

Lim $_0$ F_t s = F_t ds from A to B

This is normally written F_t ds where A and B are the end points of the curve,
$_{AB}$

or as F_t ds where the curve c connecting A and B is defined.
$_{C}$

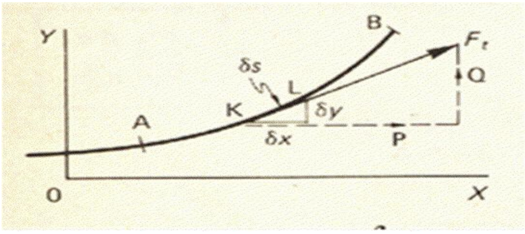
Such an integral thus formed, is called a line integral since integration is carried out along the path of the particular curve c joining A and B.

I = F_t dx = F_t ds
$_{AB}$ $_{C}$

where c is the curve y = f(x) between A(x_1, y_2) and B (x_2, y_2).
There is in fact an alternative form of the integral which is often useful, so let us also consider that.

Alternative form of a line integral

It is often more convenient to integrate with respect to x or y than to take arc length as the variable.
If F_t has a component
P in the x-direction Q in the
y-direction
then the work done from K to L can be stated as P x + Q y



$_{AB}\, F_t\, ds = \quad _{AB}\, (P\, dx + Qdy)$

where P and Q are functions of x and y. In general then, the line integral can be expressed as

$$I = \quad_{C} F_t\, ds = \quad_{C} (P\, dx + Qdy)$$

where c is the prescribed curve and F, or P and Q, are functions of x and y.

Make a note of these results then we will apply them to one or two examples.

LINE INTEGRAL

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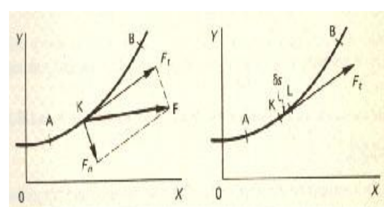

then

$$\text{Lim}_{0} \quad F_t \quad s = \quad F_t\, ds \text{ from A to B}$$

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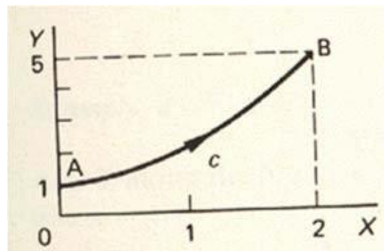
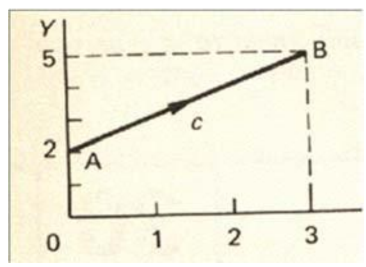
Example 1:

Evaluate (x + 3y) dx from A (0, 1) to B (2, 5)
c along the curve y = 1 + x^2.



Solution: The line integral is of the form (P dx + Qdy) where, in this case, Q = 0 and c
C
is the curve y = 1 + x^2.

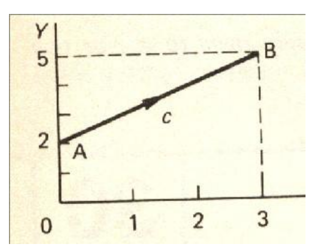
It can be converted at once into an ordinary integral by substituting for y and applying the appropriate limits of x.



I = c (Pdx+Qdy) = c (x+3y)dx = $(x+3+3x^2)dx$ (limits 2, 0)

= $\frac{x^2}{2}+3x+x^3 \Big|_0^2 = 16$

Example 2 Evaluate I = $(x^2 + y)$ dx + (x y^2)dy from A (0, 2) to B (3, 5) along the curve y = 2 + x.
C



Solution: I = (Pdx + Qdy)
C

P = x^2 + y = x^2 + 2 + x = x^2 + x + 2
Q = x y^2 = x (4+4x+x^2) = (x^2+3x+4)
Also y = 2 + x
dy = dx and the limits are x=0 to x=3

$x^2 \quad 2x \Big|_0^3 = 9 \quad 6 = 15$

$$I = \int_0^3 \{(x^2+x+2)\,dx \quad (x^2+3x+4)\,dx\} = \int_0^3 (2x+2)\,dx =$$

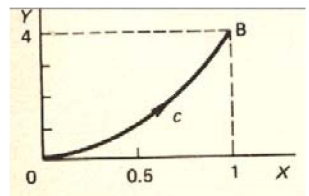
Example 3

Evaluate $I = \int_C \{(x^2+2y)dx + xydy\}$ from O(0, 0) to B(1, 4) along the curve $y=4x^2$.

Solution: In this case, c is the curve $y = 4x^2$.

$dy = 8x\,dx$

Substitute for y in the integral and apply the limits.



$$I = \int_C \{(x^2+2y)\,dx+xydy\}$$

also $x^2 + 2y = x^2 + 8x^2 = 9x^2$; $\quad xy = 4x^3$

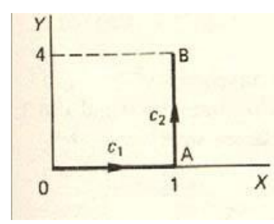
$$I = \int_0^1 9x^2\,dx \quad x \quad 4x^2 \quad 8x\,dx = \int_0^1 \{9x^2dx+32x^4dx\} = \frac{47}{5} = 9.4$$

They are all done in very much the same way.

Example 4

Evaluate $I = \int_C \{(x^2 + 2y)\,dx + xydy\}$ from O(0, 0) to A (1, 0) along the line

y = 0 and then from A (1, 0) to B (1, 4) along the line x = 1.



Solution: (i) OA : c_1 is the line $y = 0$ $\quad dy = 0$.

Substituting $y = 0$ and $dy = 0$ in the given integral gives.

$$I_{OA} = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

(ii) AB: Here c_2 is the line $x = 1$ $\quad dx=0$

$I_{AB} = 8$

$$\text{For } I_{AB} = \int_0^4 \{(1 + 2y)\,(0) + ydy\} = \int_0^4 ydy = \left[\frac{y^2}{2}\right]_0^4 = 8$$

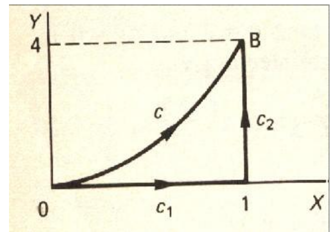
0 0

Then $I = I_{OA}+I_{AB} = \frac{1}{3} + 8 = 8\frac{1}{3}$ $\quad I = \frac{25}{3} = 8\frac{1}{3}$

If we now look back to Example 3 and 4 just completed, we find that we have evaluated the same integral between the same two end points, but along different paths of integration. If we combine the two diagrams, we have where c is the curve $y = 4x^2$ and $c_1 + c_2$ are the lines $y = 0$ and $x = 1$. The result obtained were

$I_c = 9\frac{2}{3}$ and $I_{c1+c2} = 8\frac{1}{3}$

Remark: The integration along two distinct paths joining the same two end points does not necessarily give the same results.



Properties of line integrals

1. F ds = {P dx + Q dy}

C C

2. F ds = F ds and {P dx+Q dy} = {P dx+Q dy}

AB BA AB BA

i.e. the sign of a line integral is reversed when the direction of the integration along the path is reversed.

3. (a) For a path of integration parallel to the y-axis, i.e. $x = k$, $dx = 0$

P dx = 0 $\quad I_C =$ Q dy.

C C

(b) For a path of integration parallel to the x-axis, i.e. $y = k$, $dy = 0$.

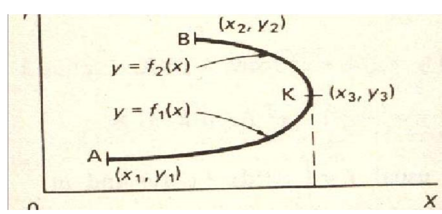
Q dy=0 $\quad I_C =$ P dx.

C C

4. If the path of integration c joining A to B is divided into two parts AK and KB, then $I_c = I_{AB} = I_{AK} + I_{KB}$.

5 .If the path of integration c is not single valued for part of its extent, the path is divided into two sections.

$y = f_1(x)$ from A to K, $y = f_2(x)$ from K to B

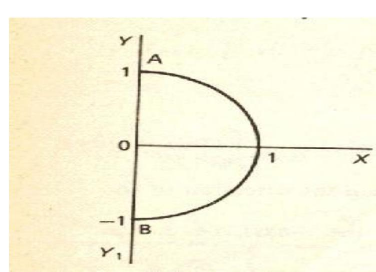


6. In all cases, the actual path of integration involved must be continuous and singlevalued.

Example 5 Evaluate I = (x + y) dx from A(0, 1) to B (0, 1) along the semi-circle $x^2+y^2=1$

C

for $x \geq 0$.

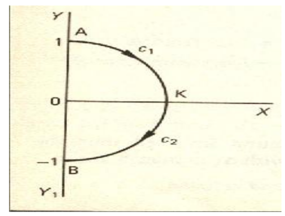


Solution: The first thing we notice is that the path of integration c is not single-valued For any value of x, y = $\sqrt{1 \quad x^2}$. Therefore, we divided c into two parts

(i) y = $\sqrt{1 \quad x^2}$ from A to K (x=0 to x=1)

(ii) y = $\sqrt{1 \quad x^2}$ from K to B (x=1 to x=0)

As usual, I = $_{C}$ (Pdx + Qdy) and in this particular case, Q = 0



$$I = {}_{C}\ Pdx = {}_{0}^{1}\ (x+\sqrt{1 \quad x^2})\ dx + {}_{1}^{0}\ (x \sqrt{1 \quad x^2}\ )\ dx$$

$$= {}_{0}^{1}\ (x + \sqrt{1 \quad x^2} \quad x + \sqrt{1 \quad x^2}\ dx = 2\ {}_{0}^{1} \sqrt{1 \quad x^2}\ dx$$

Now by trigonometric substitution, put x = sin

dx = cos d and $\sqrt{1 \quad x^2}$ $\sqrt{1 \quad \sin^2}$ $\sqrt{\cos^2}$ cos

Limits : x = 0, = 0; x = 1, $= \overline{2}$

$$I = 2 \ {}_{0}^{1} \sqrt{1 \quad x^2}dx$$

$$= 2 \ {}_{0}^{/2} \cos \quad \cos \quad d = 2 \ {}_{0}^{/2} \cos^2 \quad d = \ {}_{0}^{/2} (1+\cos 2 \quad)d = \ + \frac{\sin}{2}\ {}^{/2}$$

$2\ \overline{2}$ $_0 = \overline{2}$